

A Basic Rational Expectations Model

The model presented below draws on Bennett T. McCallum, "Rational Expectations and Macroeconomic Stabilization Policy: An Overview." *Journal of Money, Credit and Banking*. 12:4pt2 (Nov. 1980), 716-746. This paper reviews of the [then] new rational expectations methodology, which was starting to sweep macroeconomics. Overlapping generations models are one flavor, with richness added by determining output in a Solow neoclassical growth framework but simplified in another direction because they include neither money nor prices. The model presented below instead eliminates such supply-side dynamics: there is neither a savings - investment mechanism nor demographic change nor technical change. However, output is not static because the economy only gradually returns to equilibrium following random shocks to the system.

Core Model

The foundation is the following system of 4 equations in 4 unknowns: y_t , i_t , p_t and m_t . In principle it should be easy to solve, but the stochastic and expectational elements present challenges.

$$\begin{aligned} (1) \quad y_t &= a_0 + a_1[i_t - E_{t-1}(p_{t+1} - p_t)] + v_{1t} . & a_1 < 0. \\ (2) \quad m_t - p_t &= c_0 + c_1 y_t + c_2 i_t + v_{2t} . & c_1 > 0 \text{ and } c_2 < 0. \\ (3) \quad y_t &= \alpha_0 + \alpha_1[p_t - E_{t-1}(p_t)] + \alpha_2 y_{t-1} + u_t . & \alpha_1 > 0 \text{ and } 1 > \alpha_2 \geq 0 \\ (4) \quad m_t &= \mu_0 + \mu_1 m_{t-1} + \mu_2 y_{t-1} + e_t . & 0 < \mu_1 < 1 \text{ and } \mu_2 < 0. \end{aligned}$$

The final v_{1t} , v_{2t} , u_t and e_t are random variables with a normal distribution of mean 0. In addition, all of the variables y , m and p are in log form. Among other things, that (1) lets us keep everything linear (real money M/P becomes $m - p$) and (2) means that if we take a derivative we get the rate of change: if $y = \log Y$ then $\partial y / \partial t = \Delta Y / Y =$ growth rate of GDP.

The Economics

That expectations are central to macroeconomics ought not be news. Investment and savings are by definition decisions made across time; their magnitude hinges upon our beliefs about the future. So how expectations are formed and how to incorporate them into models is central to macro. Prior to "RE" models that was done poorly, for example by including a trend variable. However, not even naïve economic agents assume tomorrow will look just like yesterday.

We know that there is "noise" in the economy, particularly if we set out to construct formal models. In order to do that we have to draw boundaries: endogenous variables addressed inside our model and exogenous variables that lie outside it. Whatever model we write down will include gross simplifications while the values we pick for exogenous variables won't be precise. We reflect such incompleteness and uncertainty (though they are really different things) as a stochastic error term.

Finally, macroeconomics is about general equilibrium, in that we incorporate feedback effects among variables. In the simple "circular flow" model of Principles, final demand affects income which feeds back into final demand (that is our AS-AD multiplier). Of course we can't have everything determined in the model, because it would be too complex and would require too many special

assumptions about the interrelationships among variables; the larger the model, the more that errors cumulate. So while they remain “small” modern macro models contain multiple feedback relationships, multiple endogenous variables. Macro thus remains very different from the partial equilibrium world of microeconomics in which supply and demand are independent.

To sum up our, models are (1) dynamic, (2) stochastic and (3) general equilibrium, with (4) the formal inclusion of (rational) expectations. Modern macro revolves around the construction and simulation of DGSE (dynamic stochastic general equilibrium) models. The "engine" common to all such models contains four equations, an aggregate supply equation, an aggregate demand equation, a money supply equation, and a money demand equation. (The long-run OLG growth models we've already seen are a side branch of modern macro model, in that they exclude the price and employment mechanisms important in the short run.)

So here is what these equations actually represent:

$$(1) \quad \mathbf{y}_t = \mathbf{a}_0 + \mathbf{a}_1[\mathbf{i}_t - \mathbf{E}_{t-1}(\mathbf{p}_{t+1} - \mathbf{p}_t)] + \mathbf{v}_{1t}. \quad \mathbf{a}_1 < 0.$$

Demand is a function of a standard level of demand \mathbf{a}_0 , a stochastic component \mathbf{v}_{1t} , and the expected real interest rate (the current nominal rate \mathbf{i}_t less the expected increase in prices $\mathbf{p}_{t+1} - \mathbf{p}_t$). We don't include it explicitly, but in the background investment is affected negatively by a higher real interest rate, and the level of investment must be chosen one period in advance. This is similar to the downward-sloping AD curve encountered in Principles (Econ 102).

$$(2) \quad \mathbf{m}_t - \mathbf{p}_t = \mathbf{c}_0 + \mathbf{c}_1\mathbf{y}_t + \mathbf{c}_2\mathbf{i}_t + \mathbf{v}_{2t}. \quad \mathbf{c}_1 > 0 \text{ and } \mathbf{c}_2 < 0.$$

This is a money demand equation, in which the focus is on real money (remember that in our notation $\log(M_t/P_t) = m_t - p_t$). As per normal money-and-banking models, the demand for (real) money is higher when GDP is higher, and is lower when the nominal interest rate is higher.

$$(3) \quad \mathbf{y}_t = \mathbf{\alpha}_0 + \mathbf{\alpha}_1[\mathbf{p}_t - \mathbf{E}_{t-1}(\mathbf{p}_t)] + \mathbf{\alpha}_2\mathbf{y}_{t-1} + \mathbf{u}_t. \quad \mathbf{\alpha}_1 > 0 \text{ and } 1 > \mathbf{\alpha}_2 \geq 0$$

This is our supply-side function, *cf.* the AS curve in Principles. When the prices that suppliers see are higher than expected they increase output. In addition, we assume that the economy evolves slowly so that GDP reverts gradually back to the base $\mathbf{\alpha}_0$ level after a positive or a negative shock. The model will thus exhibit a business cycle. This sort of lag is called an "accelerationist" model, and historically was used in an investment equation to incorporate the response of investment to past investment or past GDP.

$$(4) \quad \mathbf{m}_t = \mathbf{\mu}_0 + \mathbf{\mu}_1\mathbf{m}_{t-1} + \mathbf{\mu}_2\mathbf{y}_{t-1} + \mathbf{e}_t \quad 0 < \mathbf{\mu}_1 < 1 \text{ and } \mathbf{\mu}_2 < 0.$$

This is our Taylor Rule for how the central bank responds to economic performance. Money supply should be adjusted gradually, hence depends upon past money, while past high GDP should lead to "tight" money [and, via the interaction with equation (2), higher nominal interest rates].

To see a bit more of what the model looks like, assume that we are at an equilibrium with stable prices so that $\mathbf{p}^* = \mathbf{p}_{t+1} = \mathbf{p}_t$, while we similarly assume that we have constant \mathbf{y}^* . In equation (3) the price term drops out since $\mathbf{E}(\mathbf{p}) = \mathbf{p}$. We then have $\mathbf{y}^* = \mathbf{y}_t = \mathbf{\alpha}_0 + \mathbf{\alpha}_2\mathbf{y}_{t-1} = \mathbf{y}_t = \mathbf{\alpha}_0 + \mathbf{\alpha}_2\mathbf{y}^*$, which means that $\mathbf{y}^* = \mathbf{\alpha}_0 / (1 - \mathbf{\alpha}_2)$. We can similarly solve for \mathbf{i}^* and \mathbf{m}^* . If we start away from the equilibrium we can confirm that \mathbf{y}_t would gradually converge back to \mathbf{y}^* .

Solution method

However what we have is a stochastic system that will tend back towards an equilibrium, but is not itself a steady-state equilibrium. So how do we solve this system to see (for example) what monetary policy can accomplish in the face of a negative shock.

In effect at time t we have a system of four linear equations in four unknowns: y_t , m_t , i_t and p_t . However, it is a dynamic stochastic system, so while we at base we rely upon basic algebra, we have to address the stochastic component. That requires use to deal with the expectations operator $E_t(\cdot)$. We assume that decision makers know the model and are rational, in that they have expectations of what the key variable y_t will be and make decisions on that basis. However, they will normally be wrong because of the normally distributed error terms that enter into three of the four equations.

Now "history is history" with variables known contemporaneously. So the expectations of $E_t(y_t) = y_t$. Likewise the expectations of an expectation are just the expectation: $E_t(E_t(y_{t+1})) = E_t(y_{t+1})$. The presence of noise, however, in the v_t , u_t and e_t terms in our four basic equations mean that our expectations are likely not met, even if (with mean = 0) they are accurate on average. Because the equations have variables in common, a deviation in (say) v_{1t} in equation (1) from the mean of 0 will change the value of y_t and this in turn will interact with the other equations and thus will change the values of other variables.

Step 1: eliminate "i" by basic algebra with (1) and (2), the only equations that include "i" to get equation (A-1) with y_t on the left.

Step 2: solve for "p". use (A-1) from step 1, which and combine with (3) to eliminate y_t (but not y_{t-1}). This gives us (A-2) with p_t on the left.

Step 3: take $E_{t-1}(\cdot)$ of the above result (A-2) to get (A-2a), then subtract that from (A-2). this eliminates a bunch of terms and get (A-3).

Step 4: do the same with equation (4) to get (A-4) $m_t - E_{t-1}(m_t) = e_t$ which we reinsert into (A-3a) to simplify further. to get (A-3a).

Step 5: insert (A-3a) into 3 which lets us eliminate all the expectations terms. We then get an equation for the level of y_t . The level is (1) stochastic and (2) depends on history which is the outcome based on past "draws" of the random error terms. We'll look at it more once we get ther.

Step 1: equate (1) and (2) to eliminate i_t . This is straightforward algebra:

$$(1) \quad y_t = a_0 + a_1[i_t - E_{t-1}(p_{t+1} - p_t)] + v_{1t} \text{ so we have}$$

$$(1a) \quad i_t = 1/a_1[y_t - a_0 + a_1[E_{t-1}(p_{t+1} - p_t)] - v_{1t}] \text{ while similarly from}$$

and from

$$(2) \quad m_t - p_t = c_0 + c_1 y_t + c_2 i_t + v_{2t} \text{ we get}$$

$$(2a) \quad i_t = 1/c_2[m_t - p_t - c_0 + c_1 y_t + v_{2t}].$$

So setting (1a) = (2a) gives us

$$(1b) \quad 1/c_2[m_t - p_t - c_0 + c_1 y_t + v_{2t}] = 1/a_1[y_t - a_0 + a_1[E_{t-1}(p_{t+1} - p_t)] - v_{1t}]$$

Reducing all of this with y_t on the left gives us:

$$(A-1) \quad y_t = \beta_0 + \beta_1(m_t - p_t) + \beta_2 E_{t-1}(p_{t+1} - p_t) + v_t. \quad \beta_1 > 0, \beta_2 > 0.$$

Here the β coefficients are ratios of combinations of the a and c coefficients. We need not go through the tedium of the calculations; please take my word that it is possible to show that the two economically relevant β coefficients can indeed be shown to turn out to be positive. In addition,

$$(A-1b) \quad v_t = (c_2 v_{2t} - a_1 v_{1t}) / (a_1 c_1 + c_2)$$

Since v_t is the (weighted) sum of two normally distributed stochastic variables with mean 0 it will also be normally distributed with mean 0.

Step 2: equate (3) and (A-1) to eliminate y_t . This time I skip most of the details:

$$\alpha_0 + \alpha_1[p_t - E_{t-1}(p_t)] + \alpha_2 y_{t-1} + u_t = \beta_0 + \beta_1(m_t - p_t) + \beta_2 E_{t-1}(p_{t+1} - p_t) + v_t.$$

$$(A-2) \quad p_t = 1/(\alpha_1 + \beta_1) [(\beta_0 - \alpha_0) + \beta_1 m_t + \alpha_1 E_{t-1}(p_t) - \alpha_2 y_{t-1} + \beta_2 E_{t-1}(p_{t+1} - p_t) + v_t - u_t].$$

Step 3a: take $E_{t-1}(\bullet)$ of both sides of (A-2). Remember that $E_{t-1}u_t = E_{t-1}v_t = 0$ since these are normally distributed random variables with mean 0. Likewise $E_{t-1}(E_{t-1}p_t) = E_{t-1}p_t$ and $E_{t-1}\alpha_i = \alpha_i$ since the expectation of a constant is (accurately!) the constant. Again, $E_{t-1}y_{t-1} = y_{t-1}$ since contemporaneous variables are known.

$$(A-2a) \quad E_{t-1}p_t = 1/(\alpha_1 + \beta_1) [(\beta_0 - \alpha_0) + \beta_1 E_{t-1}m_t + \alpha_1 E_{t-1}p_t - \alpha_2 y_{t-1} + \beta_2 E_{t-1}(p_{t+1} - p_t)].$$

Step 3b: we subtract (A-2a) from (A-2) to eliminate the constants and the $E_{t-1}p_t$ and the y_{t-1} variables:

$$(A-3) \quad p_t - E_{t-1}p_t = 1/(\alpha_1 + \beta_1) [\beta_1(m_t - E_{t-1}m_t) + v_t - u_t].$$

Step 4a: we take $E_{t-1}(\bullet)$ of the m_t equation (4) and subtract from (4). But the central bank fixes the money supply on the basis of past variables so $E_{t-1}(m_t) = \mu_0 + \mu_1 m_{t-1} + \mu_2 y_{t-1}$ hence:

$$(A-4) \quad m_t - E_{t-1}m_t = e_t.$$

Step 4b: substitute (A-4) in (A-3a) to get (A-3)

$$(A-3a) \quad p_t - E_{t-1}p_t = 1/(\alpha_1 + \beta_1) [\beta_1 e_t + v_t - u_t].$$

Step 5: substitute (A-3a) into (3) and reduce.

$$(A-5) \quad y_t = \alpha_0 + \alpha_2 y_{t-1} + w_t.$$

Now w_t is again a weighted average of normally distributed random variables and so is itself just another normally distributed random variable $w_t = [\alpha_1 v_t + \beta_1 u_t + \alpha_1 \beta_1 e_t] / (\alpha_1 + \beta_1)$.

Wow! – think about it! In this simple model, even though we have business cycles, monetary policy is ineffective: **no policy parameter** (neither μ_0 nor μ_1 nor μ_2) **enters into the solution we obtained for y_t** . Output depends only on y_{t-1} and on the current “draws” of our normally distributed, zero mean random variables. [Since y_{t-1} likewise depends on the “draws” of our stochastic variables and on y_{t-2} , what we really end up with is that y_t follows a mean-reverting random walk process.]

To reiterate, **policy is irrelevant** and while this is a simplified model, the addition of a fiscal policy variable does not change the bottom line (but it does make the calculations *much* more tedious).

If the central bank makes a mistake and doesn't hit its monetary supply target exactly, then its actions do end up affecting GDP. But as long as the central bank behaves predictably through some sort of Taylor Rule, then in fact monetary policy accomplishes nothing. Only surprises matter, something that is unexpected (in our precise statistical definition of the term). That bottom line was of course most welcome to economists who were distrustful of government in general and the Fed in specific. (Cf. *The Monetary History of the US* [Milton Friedman & Anna Schwartz] which found that the Fed's ineptness turned the 1929 crash into the Great Depression).

Extensions

These models are not robust to “nominal rigidities” or “wedges” such as market power (monopolistic competition) that create excess demand in product markets and rigidities in labor markets that produce inflation and unemployment. These are the core of “New Keynesian” models. The “New Classical” tradition focuses on detailing financial markets or including multiple countries.

Such models continue to revolve around positing a long-run equilibrium and then iterating to find an optimal path where expectations are met, stochastically. These mean that any “shock” to the system must be matched by offsetting behavior, typically sooner rather than later. So if governments run deficits, then these shift the budgets of everyone in the economy and convergence towards the long-run equilibrium means that in the long run tax rates and in the short run savings adjust to equilibrate the demand for and supply of assets. As a result, fiscal multipliers are forced to be small, more-or-less by assumption: if debt rises, savings rise as well, so that higher government expenditures are offset by lower consumption.

Estimating these models is also difficult. In general, parameters are chosen so that they are a good fit for the recent past. However, it is not clear what represents a “good” fit, and the credibility of the long-run equilibrium used to simulate the future is seldom questioned. Of course, by almost any standard projections using macroeconomic models of almost any flavor are not very accurate; economists have no crystal ball. Numbers of well-known macroeconomists however behave as though they have one: rational expectations are in effect build a very specific prediction of the future into models. We'll read more thereon, by those involved in constructing such models and those critical of them as presuming the results, that is, as serving not as science but merely as a reflection of political priors.